MSDL presentation

Complex Systems: Ideas from Physics

Indrani A. Vasudeva Murthy

McGill

Modelling, Simulation and Design Lab (MSDL)

School of Computer Science, McGill University, Montréal, Canada

Overview

- What are complex systems ?
  - Examples; Common Characteristics.
  - Disorder to Order; Scale Invariance, fractals and power laws.
  - Critical Phenomena; self-organization and emergent behaviour.
  - Simplicity and Complexity; equilibrium and non-equilibrium.
- Self Organized Criticality (SOC)
  - Examples and Models.
- Concluding remarks.

What are Complex Systems ?

Examples

- The universe – galaxies – stars and planetary systems.
- Weather, rainfall, earthquakes, forest fires, epidemics.
- Traffic jams, the economy and stock market.
- Biological evolution, ecoystems, social behaviour: insect colonies and swarms, flocking of birds and herding of animals; crowd behaviour; predator-prey systems.
- Pattern formation: zebra stripes, insect wings, leopard spots, sea shells.
- The human brain, the immune system.
- Organs — tissues — cells.

Common Characteristics

- A very large number of interacting units.
- The emergence of ‘order’ from ‘disorder’: collective or co-operative behaviour not obvious from the individual behaviour – leading to **self-organization** and **emergent behaviour.**
- Highly non-linear; feedback and adaptation.
- Individual units obey simple local rules. Leads to optimization, with a parallel evaluation of options.
- Hierarchical complexity - complexity on several length scales.
- **Power laws, scale invariance, self-similarity.**

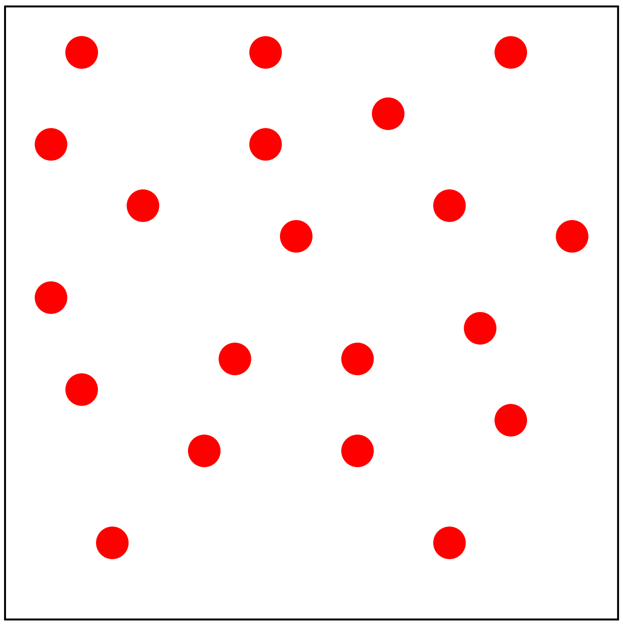
Common Characteristics

- Driven dynamical systems which are **far from thermodynamic equilibrium.**
- Computationally complex: computer models and simulation, interdisciplinary.
- Mathematical techniques: non-linear differential equations, cellular automata and difference equations, probability and stochastic theory, graph theory, game theory, genetic algorithms...
- Self-organized criticality : a possible mechanism explaining some features.

Disorder to Order

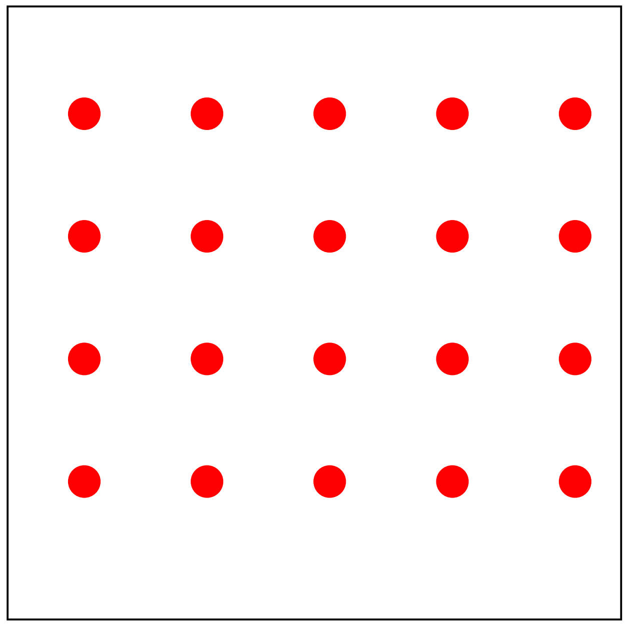
- Equilibrium **phase transitions** or **critical phenomena** from a disordered phase to an ordered phase as some parameter is varied, such as temperature.
- Disordered phase above a **critical temperature** T_C, ordered phase below it. Spontaneous symmetry breaking: state of **higher symmetry** to **lower symmetry**; higher entropy to lower entropy.
- Gas – liquid and liquid – crystal transitions: first order, discontinous.
- Paramagnet – ferromagnet; normal metal – superconductor, normal fluid – superfluid transitions; second order, continuous.
- Can define an **order parameter**: zero in the disordered phase and non-zero in the ordered phase; discontinuous or continuous.

The Gas–Liquid–Solid Transition



Disordered Gas

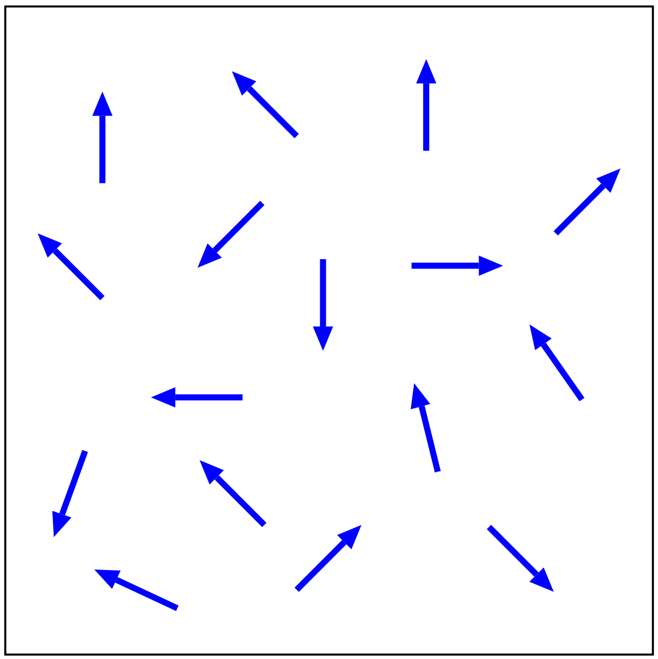
$T > T_C$



Ordered Crystal

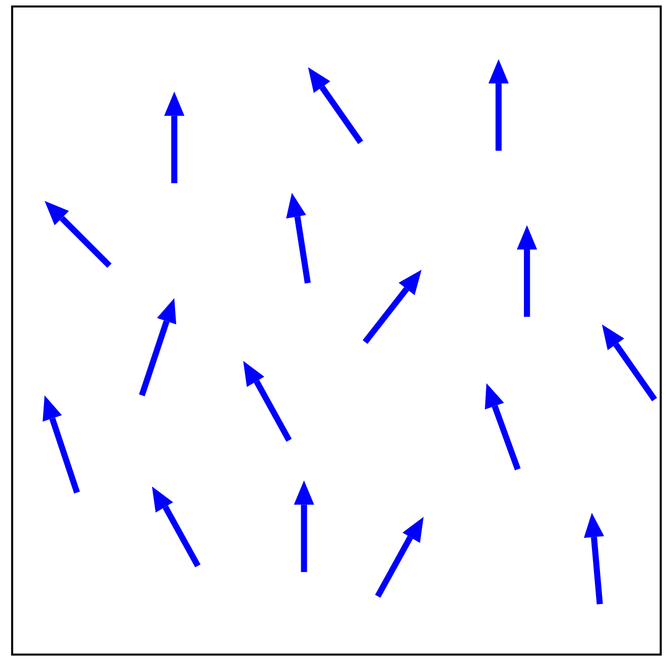
$T < T_C$

The Ferromagnetic Transition



Paramagnet

$T > T_c$



Ferromagnet

$T < T_c$

After Chaikin and Lubensky, *Principles of Condensed Matter Physics*

Scale Invariance – Self Similarity

- Scale Invariance or Self Similarity: an object ‘looks the same’ at any length scale.
- Self similar objects: **fractals**: have fractional dimensions.
- Fractals occur everywhere in nature; both spatial and temporal fractals.
- Spatial fractals: coastlines, clouds, river networks, blood vessels in the lungs, folds in the brain, . . .
- Temporal fractals: light emitted from quasars, highway traffic, sunspot activity, pressure variations in air caused by music, the height of the river Nile, . . .

Statistical Fractals: Random Walks

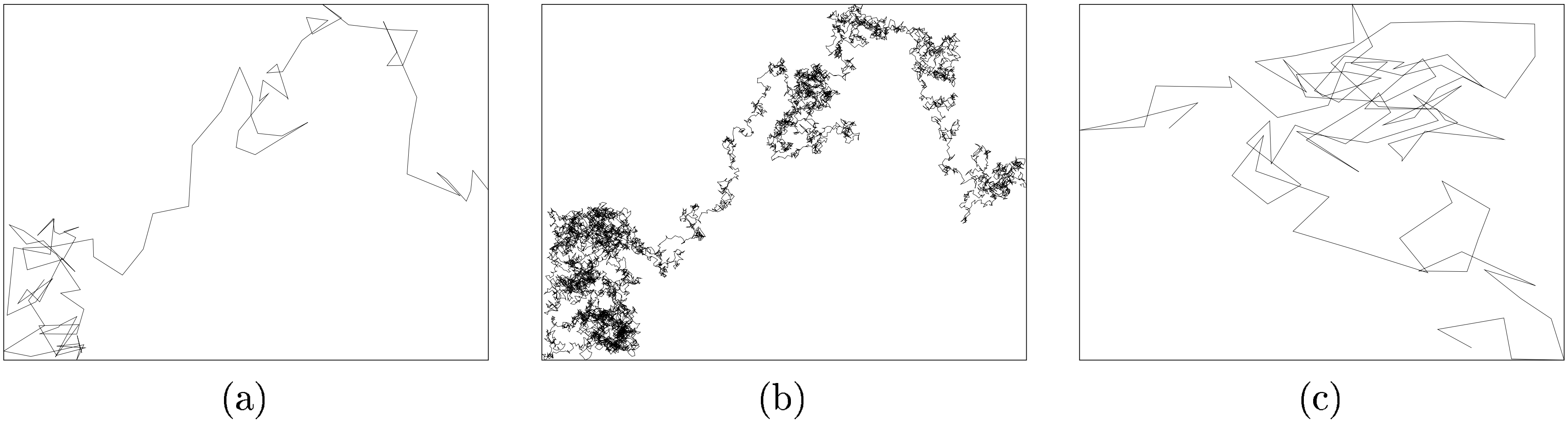


Figure 5.8 Random walks. (a) 100 big steps; (b) same as (a) except with 100 small steps between each big step; and (c) the first 100 small steps from (b)

Statistical Fractals: Stock Index

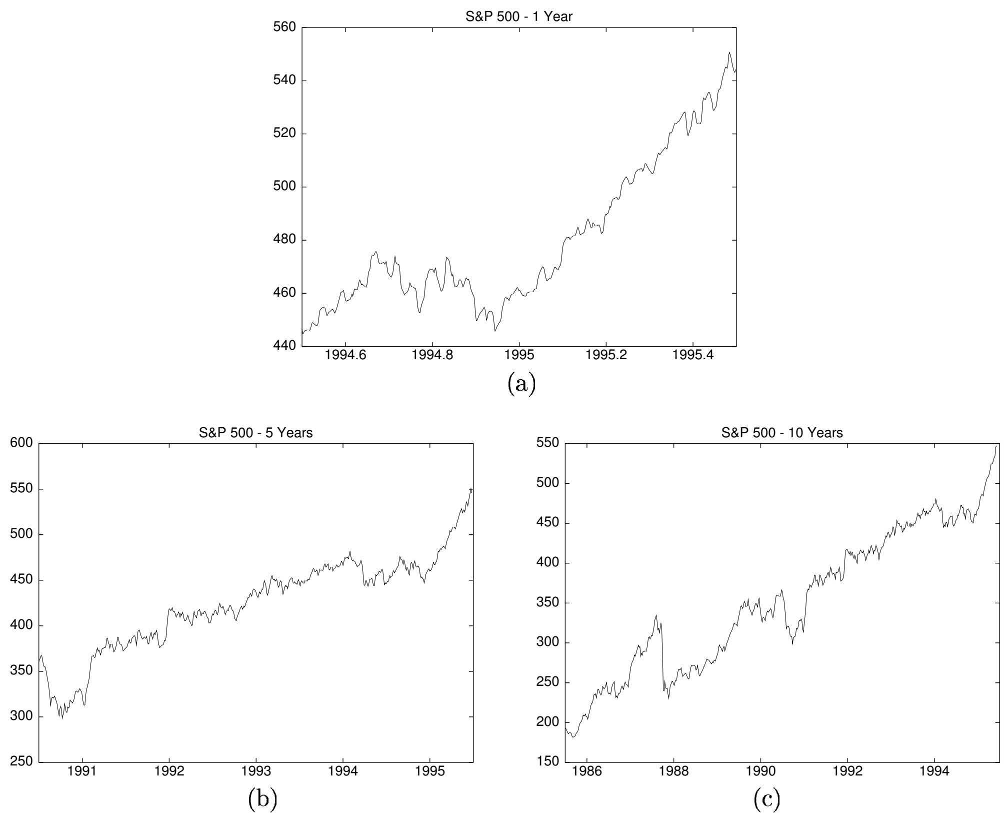


Figure 5.9 The S&P 500 stock index shown on various time scales. (a) one year, (b) five years, (c) ten years

Mathematical Fractals: The Koch Curve

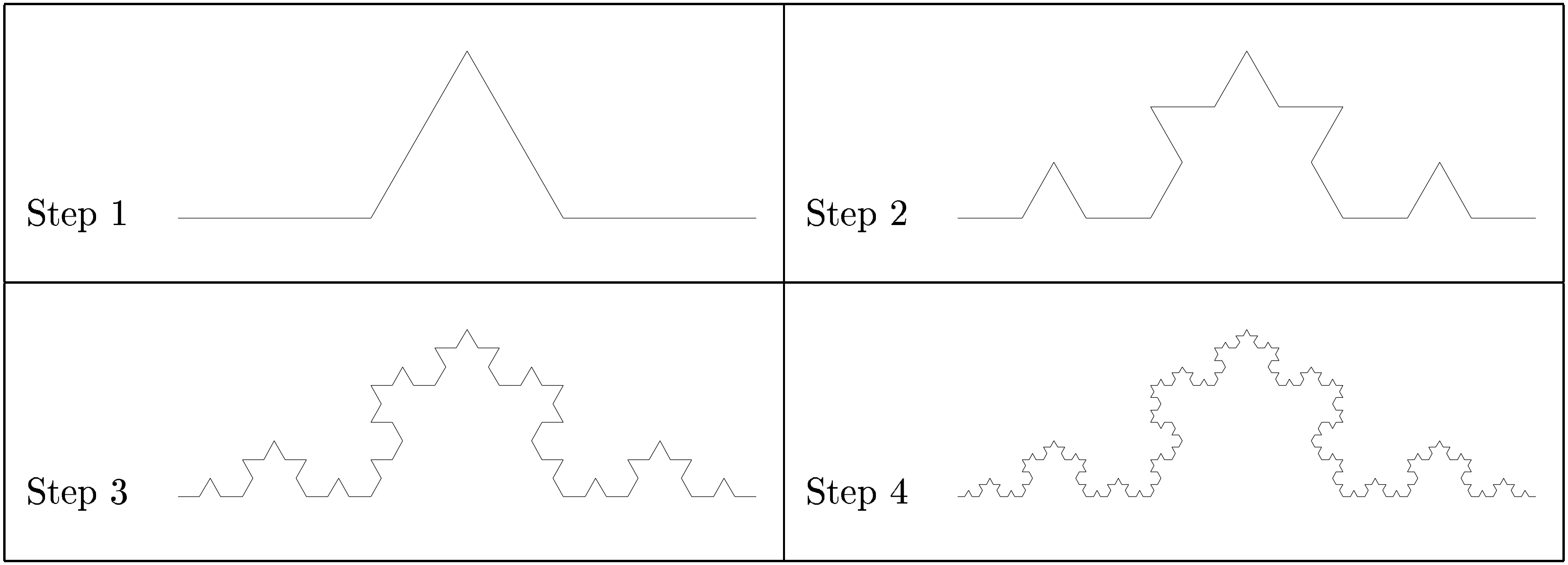


Figure 5.4 The first few steps in constructing the Koch curve

Scale Invariance – Self Similarity

- Fractals in Nature are **statistical** fractals.
- Can construct fractals - **mathematical** fractals. Examples: The Koch curve, random walks.
- The dimension of the Koch curve is $\log 4/\log 3 \simeq 1.26186$.
- An important consequence of scale invariance - occurence of **power laws.**

Scale Invariance – Power Laws

- Power law behaviour:

$$\begin{aligned} G(x) &\sim x^{-p} \\ \log G(x) &\sim -p \log x. \end{aligned}$$

$\Longrightarrow$ The plot should be a straight line, the slope gives the **exponent**.

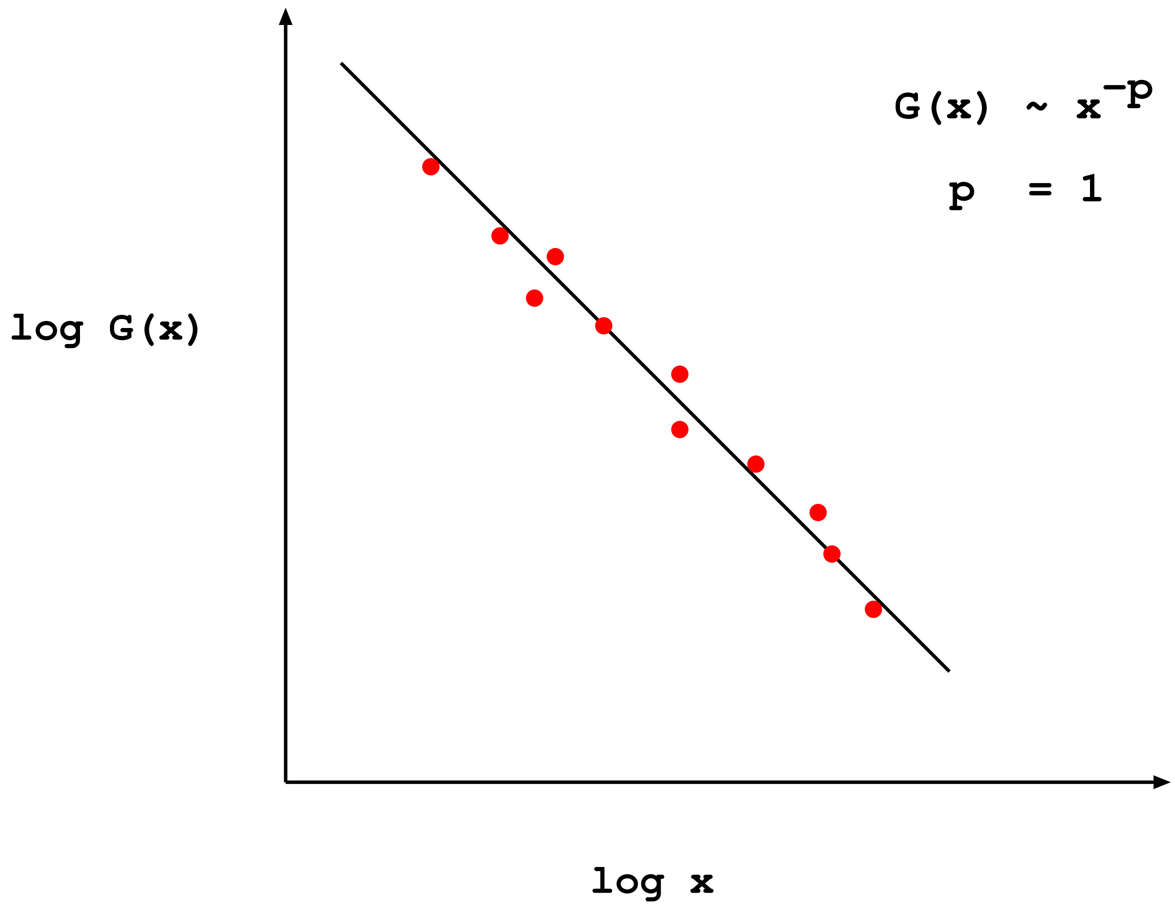
- Another way of looking at this:

$$G(bx)/G(x) = b^{-p}.$$

– Independent of x.

- A straight line: featureless, no length scale is important, or all length scales are equally important.

Power laws



Correlation functions and Power Laws

- Why are power laws important ?
- Study **co-operative behaviour**: different parts of the system interact or talk to each other: their properties or behaviour are **correlated**.
- Define **correlation functions** and look at their decay in space and time.
- For a 1-D system: for two points at (x, x'),

$$G(|x - x'|) = < m(x) m(x') >$$

.

Correlation functions and Power Laws

- For uncorrelated systems: $G(x)$ decays exponentially:

$$G(x) \simeq exp(-x/\xi)\,,$$

there is a **correlation length** ξ**.**

- For correlated systems, $G(x)$ decays only algebraically – power law:

$$G(x) \simeq x^{-p}$$

.

- In the theory of phase transitions, m is the order parameter.

Criticality

- Near criticality, the correlation length grows very large (diverges at T_C). It is the only important length scale in the system.
- The growth of the correlation length, the decay of the correlation function, and the behaviour of other quantities near T_C are all power laws: **critical exponents**.
- **Universality**: Using Renormalization Group Theory, can prove that the critical exponents only depend on the dimensionality of the system, the symmetry of the order parameter, the symmetry and range of the interaction. The exponents **do not depend** on the **details of the interactions**.
- Very different systems have identical critical exponents.

Temporal correlations: $1/f$ noise

- Look at a time-dependent signal $N(t)$: it fluctuates in time, and analyze it statistically using a correlation function (fluctuations):

$$G(\tau) = < N(t)N(t+\tau) > - < N(t) >^2.$$

Look at the decay of the fluctuation from its instantaneous value $G(0)$.

- The power spectrum $S(f)$: Fourier transform of the square of the amplitude of the signal - just cosine transform of $G(\tau)$.
- $S(f) \simeq 1/f$: $1/f$ noise - there are fluctuations of all durations - no one time scale is picked out: scale invariance in time.
- Many natural phenomena exhibit $1/f$ noise: fractals in time; visible light: ‘pink noise’.
- More generally, $1/f^{\alpha}$, with α between $(0,2]$.

Self-organization and emergent behaviour

- Classic examples: insect colonies and bird flocks. Each individual performs its own task, and collectively they achieve a totally different goal - **emergent behaviour**.
- Self-organization: no external tuning making the individuals behave collectively.
- Simulation attempts: artifical ants, termites and birds - cellular automata. A large number of interacting units. Each unit follows simple local rules.
- The system as a whole shows emergent behaviour: termites collecting wood-chips into piles, birds flocks, insect swarms, schools of fish - emergent patterns.

Flocking behaviour: Boids

- Craig Reynolds introduced generalized objects called ‘boids’: simple geometrical objects. Each boid is an individual agent following a simple set of rules, optimize individual goals.
- Rules:
  - **Avoidance:** move away from boids too close, reduce the chance of collisions.
  - **Copy:** fly in the general direction that the flock is moving in, average over the other boids’ velocities and directions (cohesion).
  - **Centre:** move towards the centre of the flock, minimize exposure to the exterior.
  - **View:** (Gary William Flake) move laterally away from any boid that blocks the view.

Flocking of birds: Boids

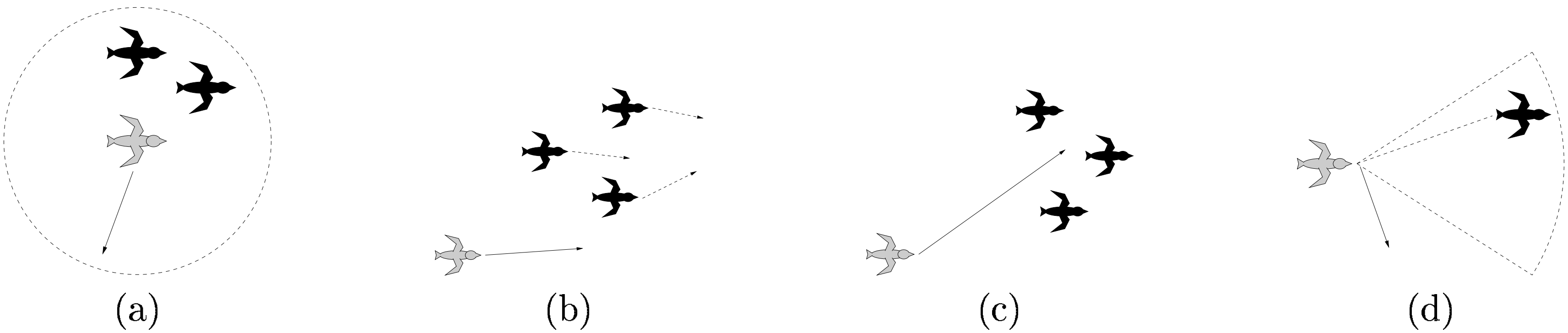


Figure 16.6 Four boid rules: (a) avoid flying too close to others; (b) copy near neighbors; (c) move towards center of perceived neighbors; (d) attempt to maintain clear view.

Flocking of birds: Boids

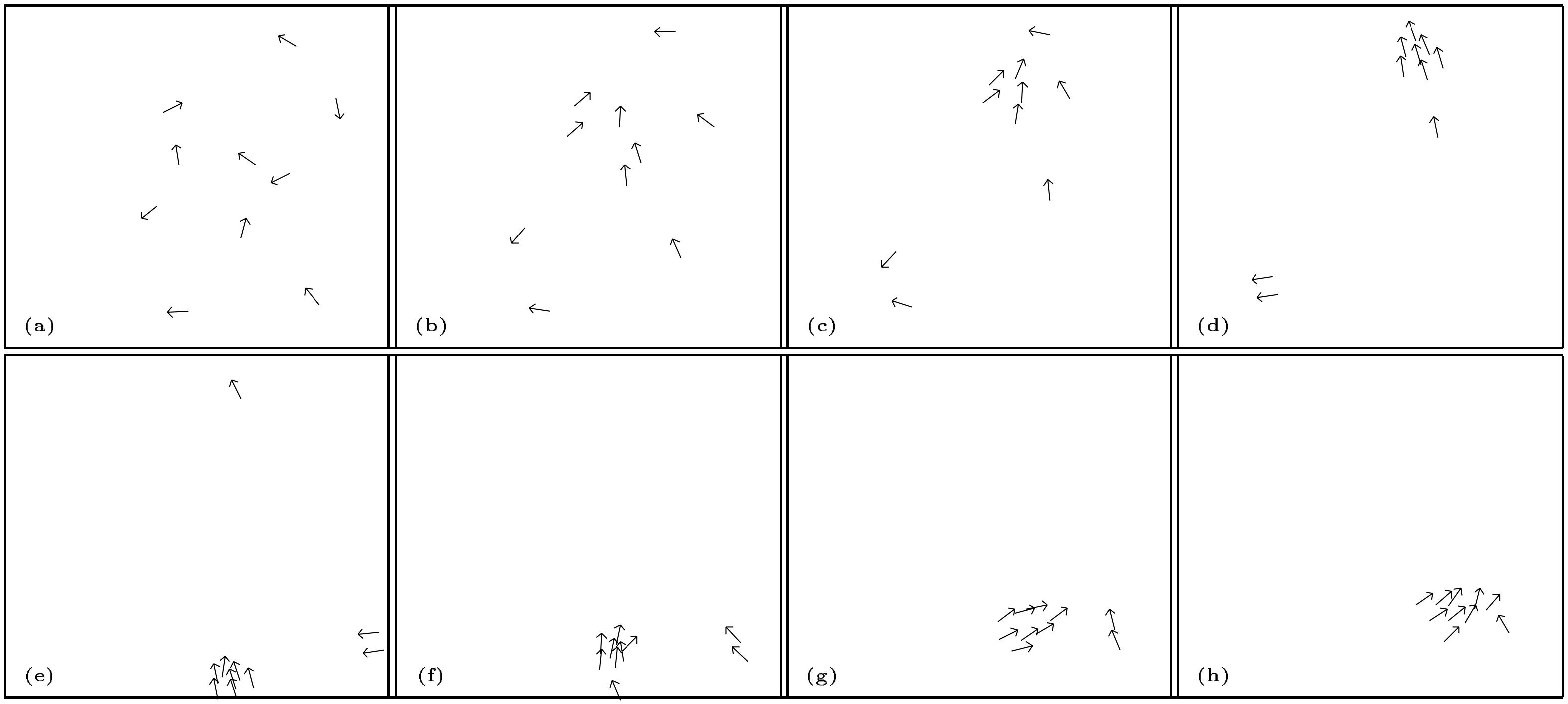


Figure 16.7 A collection of boids self-organize to form a flock

Flocking of birds: Boids

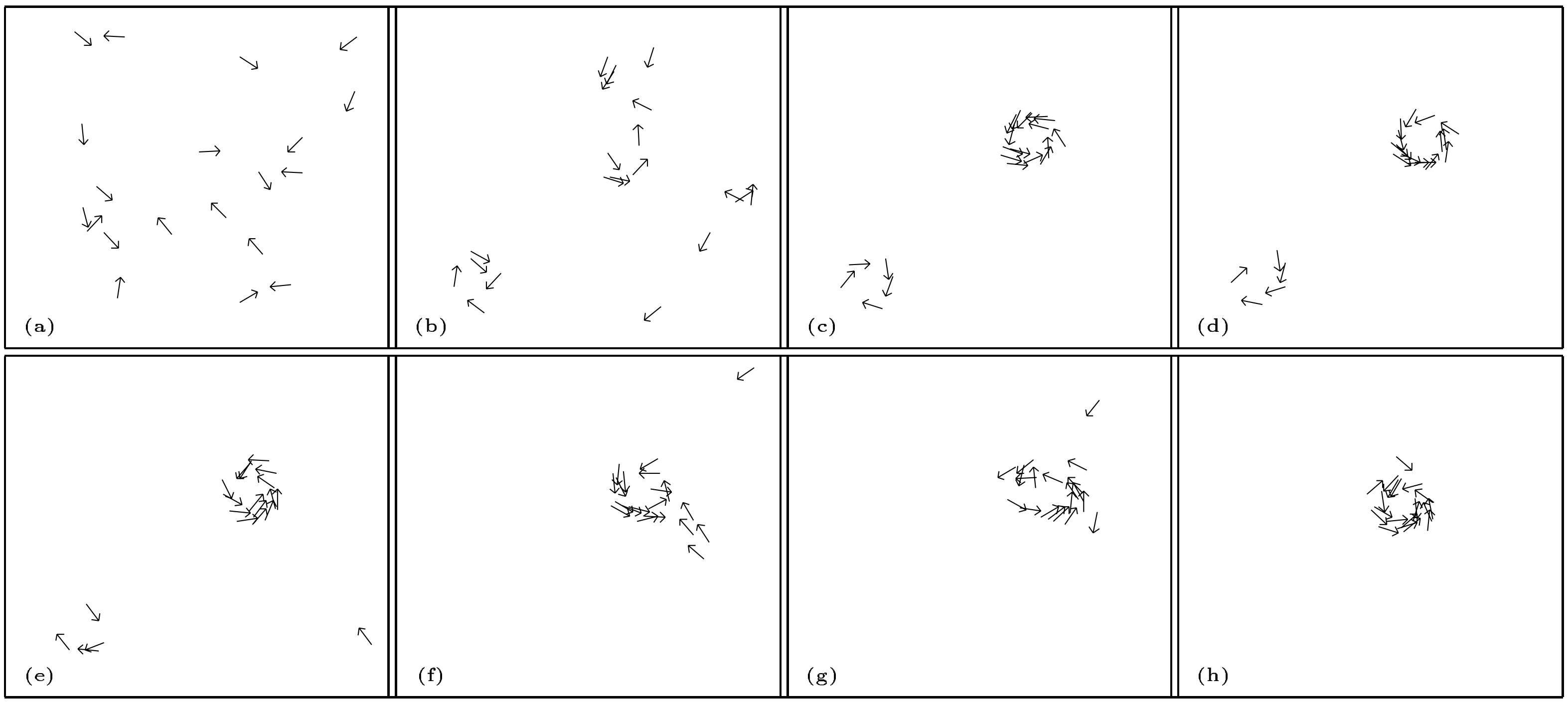


Figure 16.8 Changing the physics of the boids' universe allows for boid cycles.

Flocking of birds: Boids

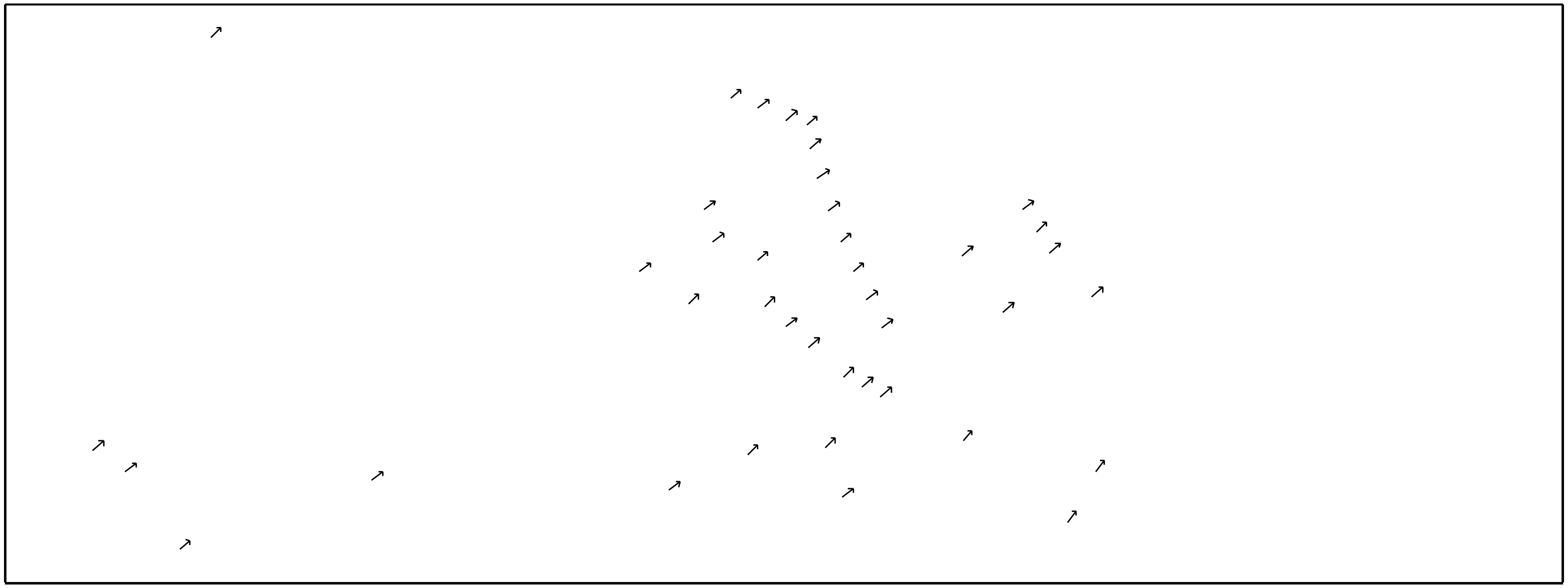


Figure 16.9 Activating the fourth boid rule promotes more realistic flocks

Flocking behaviour: Boids

- Update the velocity: a weighted sum over velocities given by the four rules: play with the weights. Also update the position.
- Intelligent behaviour! Each of the rules is a 'behavioural agent' that competes and co-operates with the other agents, ultimately yielding emergent, 'intelligent' behaviour; 'recursion' in agents: each agent made up of subagents, ...
- Physics approach (Toner and Tu, 1998): treat the birds as a fluid. Flocking represents a transition to an 'ordered phase'. The average velocity is the order parameter.

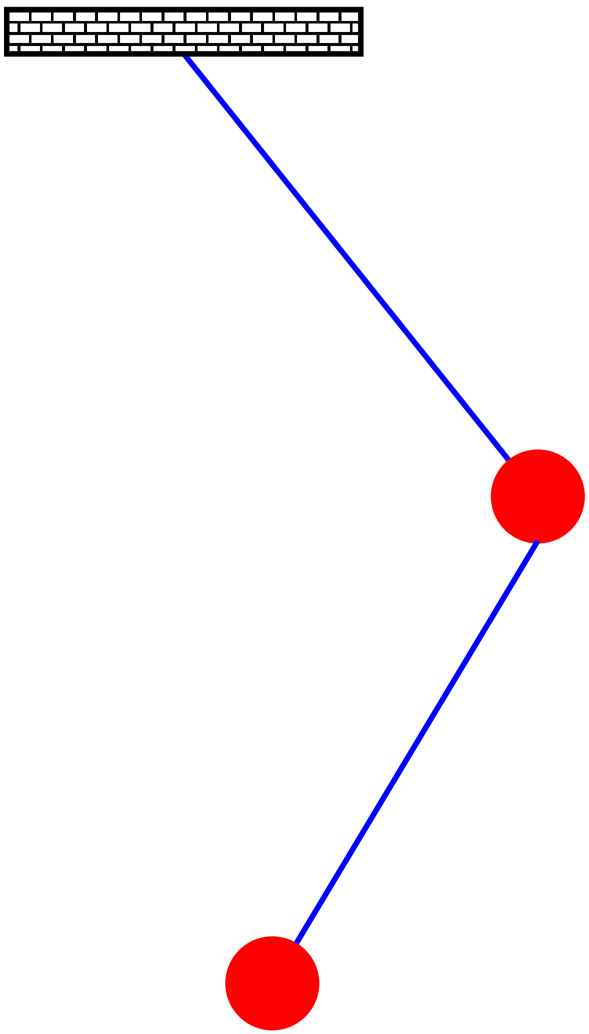
Simplicity and Complexity

- Is Nature ultimately simple ? Can describe a large number of phenomena with simple laws.
- Newton's laws of motion and gravitation, Maxwell's laws of electromagnetism - hold over a wide range of length scales.
- Reductionism: If a system has a large number of parts, break it down into smaller parts. Understand the small parts, you can understand the larger system.
- Not true for complex systems - cannot get emergent behaviour. Example, boids. 'The whole is greater than the sum of its parts'.
- Distinguish between simple systems that lead to complicated behaviour and truly complex systems.

Simplicity and Complexity

- Simple systems: write down the equations of motion, behaviour is understood.
- A double pendulum is a simple system: however can lead to **chaos**.
- Chaos: sensitivity to initial conditions; even simple deterministic systems can lead to unpredictable behaviour for a range of parameter values.
- However chaos is **not complex**.

The Double Pendulum



Equilibrium and Non-equilibrium

- How to study a system with a large number of ‘degrees of freedom’? Typically, 10^{23} molecules.
- Systems in thermodynamic **equilibrium**: do not evolve in time.
- Thermodynamics: empirical laws from measurements of macroscopic properties such as pressure, volume, temperature.
- Equilibrium **Statistical Mechanics**: Impossible to follow the motion of individual particles or entities. However can relate the observable macroscopic proerties to the **average** behaviour obtained in a statistical way.
- Equilibrium phenomena: idealized systems, but can be well approximated by controlled experiments: so well understood.

Equilibrium and Non-equilibrium

- Huge success story of this approach: critical phenomena.
- The appearance of order in critical phenomena is boring! They are now not considered complex: there is no emergent behaviour. (Are power laws emergent behaviour ?)
- Real systems in Nature are **non-equilibrium**, open systems; continuous inflow of energy, dissipation.
- The order seen, such as in pattern formation or self-organization, is **dynamical.**

Motivation for SOC

- $1/f$ noise, fractals and power laws in various natural phenomena : motivating features leading to the formulation of SOC. Per Bak: 'father' of SOC.
- Scale invariance and power laws : indicative of critical behaviour. For critical systems, there is tuning: the temperature.
- No external 'tuning' required for systems to show collective behaviour: self-organization.
- The dynamics of a complex system characterized by avalanche-like changes in the system state: long periods of stasis or quiescence, followed by avalanche-like events - a chain of events.
- The growth of the avalanches, or their size distribution, follows simple power laws.

Earthquakes

- Gutenberg-Richter law. A measure of the size of an earthquake is the energy released. Plot the number of earthquakes of a given energy against the energy on a log scale:

$$N(E) \sim E^{-B}; \quad B : [1.8, 2.2].$$

- The temporal frequency of aftershocks (Omori law): the number of aftershocks occuring after a major earthquake:

$$n(t) \sim t^{-A}; \quad A : [1, 1.5]$$

The morphology of the faults is fractal.

Biological Evolution

- Extinction events - not a gradual process.
- 'Punctuated equilibrium' (Stephen Jay Gould): long quiet periods followed by bursts of activity.
- 'Co-evolution': different species become extinct together.
- The 'size' of an extinction event is measured by the number of species becoming extinct together. Paleontological data for 600 million years (Sepkowski and Raup).
- Plot the number of genera against their lifetimes (log-log): power law distribution, with exponent 2.
- The Bak–Sneppen model of evolution (1993).

Software Evolution

- Punctuated equilibrium in software evolution (Gorshenev and Pismak, 2004).
- MOZILLA, FREE-BSD, Gnu-EMACS. Changes or modifications are avalanche-like events. Data from version control systems - CVS.
- For each change of file, count the number of lines deleted (D), number of lines added (A). Distributions $P(A)$ and $P(D)$ - power laws.
- $P(A) \sim A^{-a}$ and $P(D) \sim D^{-d}$.

  FREE-BSD : a = 1.44, d = 1.48.

  MOZILLA : a = 1.43, d = 1.47.

  EMACS : a = 1.39, d = 1.49.
- Is it SOC ?

SOC Models: The Sandpile

- Cannot really experiment with earthquakes, evolution, forest fires, ...
- The sandpile paradigm (Bak, Tang, Wiesenfield, 1987): add sand grains slowly. The slope of the pile increases. Beyond a certain slope the pile becomes unstable and there are avalanches.
- The simplest version: a square grid in two dimensions - a cellular automaton.
- A square of the grid is located at (i, j). Define a function $Z(i, j)$, the total number of sand grains on that square, or the local height of the sandpile.
- The inital state of the sandpile: for every square of the grid, assign a number between 0 and 3. This is a stable sandpile, where we have chosen the threshold to be 3.

The Sandpile Model

- Add a grain of sand randomly to this sandpile: $Z(i,j) \longrightarrow Z(i,j)+1$.
- Dynamics: 'toppling' rule: if the total number of grains exceeds the threshold $Z_C = 3$, the square topples and distributes 1 grain each to its nearest neighbours:

$$\begin{aligned} Z(i\pm 1,j) &\longrightarrow Z(i\pm 1,j)+1; \\ Z(i,j\pm 1) &\longrightarrow Z(i,j\pm 1)+1; \\ Z(i,j) &\longrightarrow Z(i,j)-4. \end{aligned}$$

- Open boundaries: the grains leave the system when the site topples.
- Initially nothing happens. After several time steps, one site may topple.

The Sandpile Model

- At some point, once a site topples, a neighbour will also topple at the next time step. This is an 'avalanche' - the toppling may continue to several orders of neighbours.
- After a long time, the sandpile reaches a stationary state, where the average height does not change, less than the threshold.
- At this SOC state, addition of a grain anywhere might lead to an avalanche of any size.
- The system never reaches the maximally stable state where the height = 3 for all sites.

The Sandpile Model (Per Bak, *How Nature Works*)

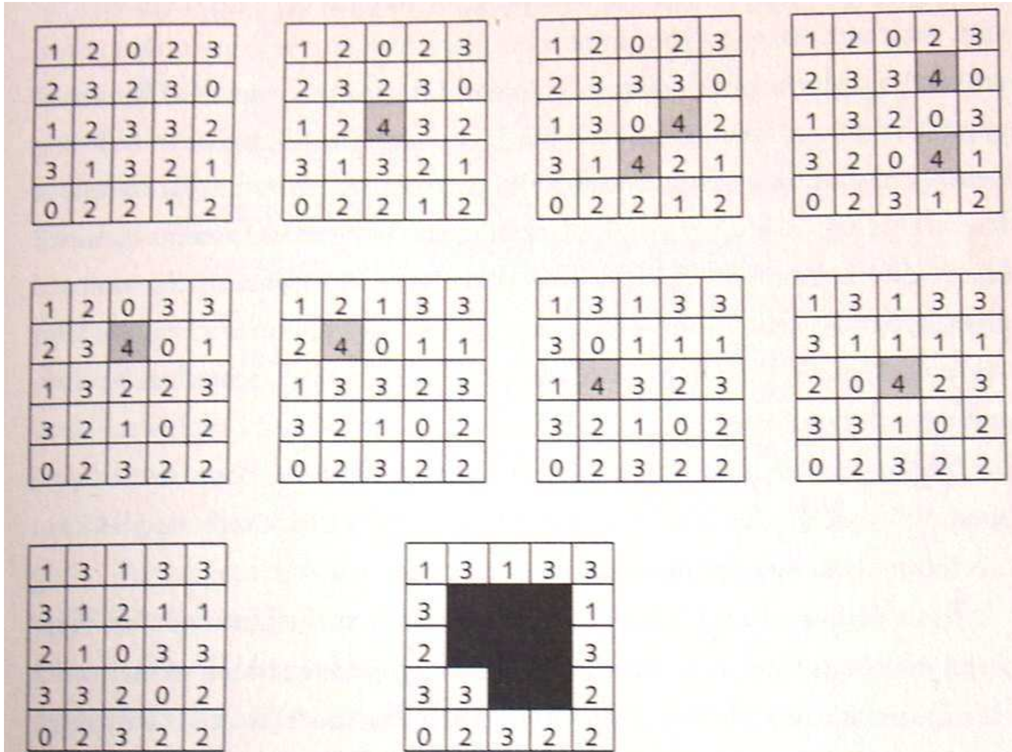


Figure 12. Illustration of toppling avalanche in a small sandpile. A grain falling at the site with height 3 at the center of the grid leads to an avalanche composed of nine toppling events, with a duration of seven update steps. The avalanche has a size $s = 9$. The black squares indicate the eight sites that toppled. One site toppled twice.

The Sandpile Model

- Number of avalanches $N(s)$ of a given size s:

$$N(s) \simeq s^{-p}. \quad p \sim 1.1$$

- The durations of the avalanches also follow a power law - not $1/f$ but $1/f^2$.
- The SOC state is robust. Vary the kind of grid; the threshold; topple by adding sand to random neighbours; increase the height by random amounts; remove the randomness: in all cases the sandpile gets into an SOC state.
- Cannot arrive at the exponent analyically: true of most SOC models.

The Sandpile Model – Theoretical Approach

- Discrete version of a diffusion process.
- The updating algorithm can be written as an 'equation of motion'.
- The height $Z(\mathbf{r},t)$ is a real, continuous variable at a random lattice position $\mathbf{r}$. It is incremented by a random amount $\eta \in [0,1]$. The threshold is Z_C.
- The update rule then becomes (time and space are discrete):

$$\begin{aligned} Z(\mathbf{r},t+1) &= Z(\mathbf{r},t)\,[1-\Theta(Z(\mathbf{r},t)-Z_C)] \\ &= +\sum_{\mathbf{r}_{nn}} \frac{1}{4} Z(\mathbf{r}_{nn},t)\,\Theta(Z(\mathbf{r}_{nn},t)-Z_C)+\eta(\mathbf{r},t) \ . \end{aligned}$$

- Here Θ is the Heaviside step-function: $\Theta(x) = \begin{cases} 0 & x < 0; \\ 1 & x \geq 0. \end{cases}$

The Sandpile Model: Continuum Limit

- Mean Field Theory: popular first attack in a theory, for large number of variables. Qualitatively correct answers.
- Replace the site variable $Z(\mathbf{r},t)$ by a 'coarse grained' value: an average over a local neighbourhood - 'mean field'.
- The equation of motion becomes:

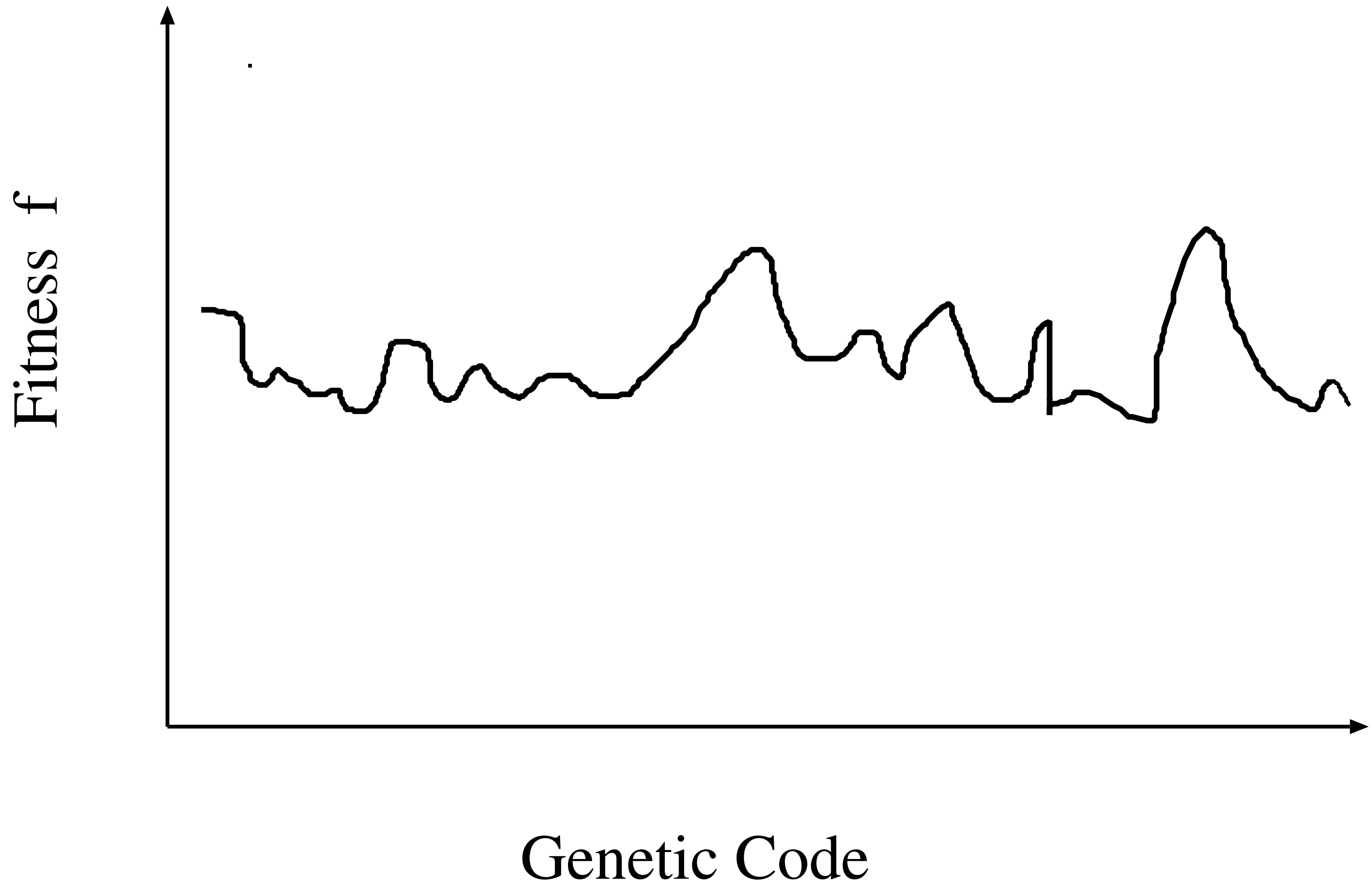
$$\frac{\partial Z(\mathbf{r},t)}{\partial t} = D\nabla^2 \left[Z(\mathbf{r},t)\,\Theta(Z(\mathbf{r},t) - Z_C)\right] + \eta(\mathbf{r},t).$$

- This is a **stochastically driven diffusion equation**, Langevin equation.
- Not easy to solve, because of the discontiuous non-linear term.
- DDRG: dynamically driven renormalization group theory.

The Bak–Sneppen Model of evolution

- A simple model of evolution of interacting species.
- Self-organizes into a critical steady state with intermittent avalanches of all sizes: punctuated equilibrium.
- A coarse grained model at the species level: the entire species is represented by a single 'fitness'.
- The fitness of a species is affected by the fitness of other species.
- The stability of each species corresponds to a 'fitness barrier'.
- High fitness $\Rightarrow$ high barriers $\Rightarrow$ stable states.
- Low fitness $\Rightarrow$ low barriers $\Rightarrow$ more likely to mutate.
- 'Fitness' landscape (Sewall Wright).

The fitness landscape



The Bak–Sneppen Model

- N species are arranged on a line with periodic boundary conditions: circle.
- For each species i, assign a random barrier B_i uniformly distributed in $[0, 1)$.
- The ecology is updated: at each time step t,
  - locate the site with the **lowest** barrier, mutate by assiging a new random number.
  - change the fitness of its two neighbours, with new random numbers.
- Initially, isolated events. After a long time, clusters of sites begin to mutate together: SOC state with a threshold, $B_C \sim 0.67$.

The Bak–Sneppen Model

- At criticality, the avalanche distribution; the correlation function; the interval between mutations for a given site, are all power laws.
- SOC is robust: change initial conditions, interactions: random neighbours, ...
- The critical fitness is **not** 1, but less.
- Darwinian picture: gradual evolution, survival of the fittest.
  SOC picture: elimination of the least fit.
- Dinosaur extinction is not special ! Need not invoke an external catastrophe.
- Software evolution: similar model, with ‘program fitness’.

Summary of SOC

- How real is SOC ? Where do you see SOC ? What new things have we learned ?
- Real sandpiles don't show SOC, but rice piles do. Avalanche-like events in other experiments.
- Not all power law behaviour or $1/f$ noise systems are SOC systems.
- SOC systems: Slowly Driven Interaction Dominated Threshold Systems (SDIDT: Jensen).

Summary of SOC

- Separation of time scales because of a **threshold**: a slow driving force (dropping of sand, building up of strains in the earth's crust), and faster internal relaxation between **metastable** states.
- Interaction dominated many-body systems. The slow drive is weak, interaction controls the dynamics.
- Self-organization: no external tuning, the interactions themselves provide the threshold.
- A large number of metastable states.
- Fluctuations are important.

Concluding Remarks

- Complex Systems: open problem.
- From the point of view of Physics, equilibrium systems are well understood.
- Cannot extend Statistical Mechanics to non-equilibrium systems.
- Systems with dissipation, time-dependent probabilities, dynamical order . . . : energy minimization may not be true, no unique ground state.
- SOC offers some insight into power laws and scale invariance.
- Wait for the Unified Theory of Everything !

References

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- *How nature works*, **Per Bak.**
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- *The Computational Beauty of Nature*, **Gary William Flake.**
- *Statistical Mechanics*, **Kerson Huang.**
- *Principles of Condensed Matter Physics*, **P.M.Chaikin and T.C.Lubensky.**
- Links to websites and papers at: http://msdl.cs.mcgill.ca/people/indrani/links.