An introduction to Category Theory

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March 14, 2002

Plan

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Motivation

- Computer Science: Types, Functions, Computation, Languages.
- Multi-paradigm modelling.
  - Multiformalism modelling: solving problems by analogy.
  - Metamodelling: describing “structures” (models, systems, general mathematical entities).
  - Multiple levels of abstraction.
- Its fundamental.
  - Common framework for mathematics.
  - Emphasis on transformations between structures.
  - Proposed as an alternative foundation.

Categories: different views

- Categories as algebraic/graphical structures.
- Mathematical structures as categories (monoids, preorders, ...).
- Categories as frameworks, i.e. categories of mathematical structures (categories of monoids, categories of preorders, categories of categories, categories of automata, categories of grammars, DEVS, etc).

Graphs with labelled edges

- Graphs with multiple edges between two vertices.
- A *(directed) labelled-edge multi-graph* is a 4-tuple (V, E, dom, cod) where V is a set of vertices or nodes, E is a set of edges or arcs, and $dom, cod : E \to V$.
- We write $f : a \to b$ or $a \xrightarrow{f} b$ for any $a, b \in V$ and $f \in E$ whenever $dom(f) = a$ and $cod(f) = b$.
- We say that two edges $f, g \in E$ are *composable* iff $dom(g) = cod(f)$, i.e. $f : a \to b$ and $g : b \to c$.
- Graphs with no labels on the edges, or with only one edge allowed per pair of objects are a specific case.

Categories: definition

- A *Category* is a 5-tuple $(O, A, dom, cod, \circ)$ where O is a collection of *objects*, A is a collection of *arrows* (usually called maps, or morphisms), $dom, cod : A \to O$, and $\circ : A \times A \rightharpoonup A$, satisfying the following axioms:
  - For any composable pair of arrows, the composition always exists: if $f : a \to b \in A$ and $g : b \to c \in A$, then $g \circ f : a \to c \in A$.
  - For every object a, there is an *identity* arrow, i.e. an arrow $1_a : a \to a$ s.t. for any $f : a \to b$, $f \circ 1_a = f$, and for any $g : c \to a$, $1_a \circ g = g$.
  - Arrow composition is associative: for any composable arrows $f, h, g \in A$, $f \circ (g \circ h) = (f \circ g) \circ h$.
- (Notation) If $\mathbf{C} = (O, A, dom, cod, \circ)$ is a category, then $Obj(\mathbf{C}) = O$ and $Mor(\mathbf{C}) = A$.

Commutative Diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r, "x"] \arrow[dr, "y \circ x"'] & \bullet \arrow[d, "y"] \\
 & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "g"'] \arrow[dr, "k \circ g"] & A \arrow[r, "f"] \arrow[dr, "h \circ f"] & B \arrow[d, "h"] \\
C \arrow[r, "k"] & D & D
\end{tikzcd}$$

...where $k \circ g = h \circ f$

Examples of mathematical structures as categories

- A *preorder* is a pair $(S, \sqsubseteq)$ where $\sqsubseteq \subseteq S \times S$ is a reflexive and transitive binary relation on S.
- Examples: $(\mathbf{N}, \leq)$, $(\mathbf{R}, \geq)$, $(2^A, \subseteq)$ where A is any set.

$$
\begin{tikzcd}
 & \{0,1,2\} & \\
\{0,1\} \arrow[ur] & \{0,2\} \arrow[u] & \{1,2\} \arrow[ul] \\
\{0\} \arrow[u] \arrow[ur] & \{1\} \arrow[ul] \arrow[ur] & \{2\} \arrow[ul] \arrow[u] \\
 & \emptyset \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

- Every preorder induces a category.

Examples of mathematical structures as categories

- A *monoid* is a triple $(S, \otimes, e)$ where S is a set, $\otimes : S \times S \to S$, $e \in S$, such that e is the identity of S with respect to $\otimes$, and $\otimes$ is associative.
- Examples: $(\mathbf{N}, \cdot, 1)$, $(\mathbf{R}, +, 0)$, $(2^A, \cup, \emptyset)$ where A is any set.
- Every monoid induces a category, where arrows are given by the elements of the set.

Examples of categories of mathematical structures

- A *monotonic function* is a function between two preorders that preserves the order relation, i.e. if $(S_1, \sqsubseteq_1)$ and $(S_2, \sqsubseteq_2)$ are preorders, then $f : S_1 \to S_2$ is a monotonic iff for any $x, y \in S_1$, $x \sqsubseteq_1 y$ implies that $f(x) \sqsubseteq_2 f(y)$.
- **Pre** is the category of all possible preorders as objects, and all possible monotonic functions as arrows.
- A *monoid homomorphism* if a function between two monoids that preserves the operation and the identity, i.e. if $(M_1, \otimes_1, e_1)$ and $(M_2, \otimes_2, e_2)$ are monoids, then $h : M_1 \to M_2$ is a monoid homomorphism iff for any $x, y \in M_1$, $h(x \otimes_1 y) = h(x) \otimes_2 h(y)$ and $h(e_1) = e_2$.
- **Mon** is the category of all possible monoids as objects and all possible monoid homomorphisms as arrows.

Examples of categories of mathematical structures

- **Set** is the category of all sets as objects and all functions as arrows.
- Types and functions or programs.
- Graphs and graph homomorphisms.
- Vector Spaces and Linear Transformations.
- Transition Systems and simulation form a category.
- Topological Spaces and continuous functions.
- etc...

Examples of categories of mathematical structures

- Graphs as objects and graph-transformations as arrows.
  - Let $\mathcal{G}$ be the set of all possible graphs. Let T be a graph-grammar. Define $\rightarrow_T^* \subseteq \mathcal{G} \times \mathcal{G}$, as follows: for any graphs $G_1, G_2 \in \mathcal{G}$, $G_1 \rightarrow_T^* G_2$ if G_2 is obtained from G_1 by applying any number of rules from T.
  - Take a particular graph-grammar T and a particular graph G. Then $\mathbf{C}_T(G)$ is a category where $Obj(\mathbf{C}_T(G)) \stackrel{def}{=} \{G' \in \mathcal{G} \mid G \rightarrow_T^* G'\}$ and $Mor(\mathbf{C}_T(G)) \stackrel{def}{=} \{(G_1, G_2) \mid G_1, G_2 \in \mathcal{G}, G_1 \rightarrow_T^* G_2\}$.
  - $\mathbf{C}_T$ is a category where $Obj(\mathbf{C}_T) \stackrel{def}{=} \mathcal{G}$ and $Mor(\mathbf{C}_T) \stackrel{def}{=} \bigcup_{G \in \mathcal{G}} Mor(\mathbf{C}_T(G))$.

Constructing new categories from old

- A *subcategory* of a category $\mathbf{C}$ is a category $\mathbf{D}$ where $Obj(\mathbf{D}) \subseteq Obj(\mathbf{C})$, $Mor(\mathbf{D}) \subseteq Mor(\mathbf{C})$, and composition is the same.

- The *dual* of a category $\mathbf{C}$ is a category $\mathbf{C}^{op}$ where
  $Obj(\mathbf{C}^{op}) \stackrel{def}{=} Obj(\mathbf{C})$,
  $Mor(\mathbf{C}^{op}) \stackrel{def}{=} \{f^{op} : b \to a \mid f : a \to b \in Mor(\mathbf{C})\}$, and
  $f^{op} \circ^{op} g^{op} \stackrel{def}{=} (g \circ f)^{op}$.

- The *product* of two categories $\mathbf{C}$ and $\mathbf{D}$ is a category $\mathbf{C} \times \mathbf{D}$ where $Obj(\mathbf{C} \times \mathbf{D}) \stackrel{def}{=} Obj(\mathbf{C}) \times Obj(\mathbf{D})$,
  $Mor(\mathbf{C} \times \mathbf{D}) \stackrel{def}{=} Mor(\mathbf{C}) \times Mor(\mathbf{D})$, and
  $(f_1, g_1) \circ (f_2, g_2) \stackrel{def}{=} (f_1 \circ f_2, g_1 \circ g_2)$.

The structure of categories

- An arrow $f : a \to b$ is called an *isomorphism* iff there is an arrow $g : b \to a$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$.
  - In **Set** isomorphisms are bijective functions.
  - In **Grf** isomorphisms are graph isomorphisms.
  - In **Top** they are homeomorphisms.
  - etc.
- An arrow $f : a \to b$ is called a *monomorphism* or *monic* iff for any pair of arrows $x, y : t \to a$, $f \circ x = f \circ y$ implies $x = y$.
  - In **Set** monics are injective functions.
- Elements of sets can be understood as arrows.

The structure of categories

- An *initial object* is an object that has exactly one arrow to every other object in the category. (In **Set** is the empty set.)
- Given any category that has initial objects, all of the initial objects are isomorphic.
- A *terminal object* in a category $\mathbf{C}$ is an initial object in $\mathbf{C}^{op}$. (In **Set** all singelton sets are terminal.)

The structure of categories

- Given two objects a, b, a *product* is an object (written $a \times b$) with two arrows $\pi_1 : a \times b \to a$ and $\pi_2 : a \times b \to b$ such that the following diagram commutes for any object c with arrows $f : c \to a$, $g : c \to b$ and $h : c \to a \times b$, where h is unique.

$$\begin{tikzcd}
a & a \times b \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & b \\
& c \arrow[ul, "f"] \arrow[u, dashed, "h"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

- In **Set** it represents the cartesian product. In **Pre** represents the "meet" (glb). In Logic represents conjunction, etc.

- The dual concept is called *sum*. In **Set** sums are disjoint unions. In **Pre** represents the "join" (lub).

Functors

- Maps that preserve the structure of categories.
- A *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ is a pair of functions $F_0 : Obj(\mathbf{C}) \rightarrow Obj(\mathbf{D})$ and $F_1 : Mor(\mathbf{C}) \rightarrow Mor(\mathbf{D})$ such that:
  - For any arrow $f : a \rightarrow b \in Mor(\mathbf{C})$, $F_1(f) : F_0(a) \rightarrow F_0(b) \in Mor(\mathbf{D})$ (Connectivity is preserved).
  - For any object $a \in Obj(\mathbf{C})$, $F_1(1_a) = 1_{F_0(a)}$. (Identities are preserved).
  - For any pair of arrows $f, g \in Mor(\mathbf{C})$, $F_1(g \circ_{\mathbf{C}} f) = F_1(g) \circ_{\mathbf{D}} F_1(f)$. (Composition is preserved).

Functors

$$
\begin{tikzcd}
a \arrow[r, "F_0"] \arrow[d, "f"'] & F_0(a) \arrow[d, "F_1(f)"] \\
b \arrow[r, "F_0"] & F_0(b)
\end{tikzcd}
\qquad f \xRightarrow{F_1} F_1(f)
$$

$$
\begin{tikzcd}
f \arrow[r, "F_1"] \arrow[d, "dom"'] & F_1(f) \arrow[d, "dom"'] \\
a \arrow[r, "F_0"] & F_0(a)
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, "F_1"] \arrow[d, "cod"'] & F_1(f) \arrow[d, "cod"'] \\
b \arrow[r, "F_0"] & F_0(b)
\end{tikzcd}
$$

Higher categories

- The category of categories.
  - The objects are categories.
  - The arrows are functors.
- The category of functors.
  - The objects are functors.
  - The arrows are natural transformations.
- 2-Categories: objects, arrows, and arrows between arrows.
- n-Categories: objects, arrows, and arrows between arrows, arrows between arrows between arrows, etc...

Acknowledgements

The diagrams where created with Paul Taylor's `diagram` package.

References

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